Core equivalence and welfare properties without divisible goods *

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Abstract

We study an economy where all goods entering preferences or production processes are indivisible. Fiat money not entering consumers' preferences is an additional perfectly divisible parameter. We establish a First and Second Welfare Theorem and a core equivalence result for the rationing equilibrium concept introduced in Florig and Rivera (2005a). The rationing equilibrium can be considered as a natural extension of the Walrasian notion when all goods are indivisible at the individual level but perfectly divisible at the level of the entire economy.

As a Walras equilibrium with money is a special case of a rationing equilibrium, our results also hold for Walras equilibria with money.

Keyword indivisible goods, competitive equilibrium, Pareto optimum, core.
JEL Classification: D50; D60.

Comment on the present version

This version imposes a less restrictive concept of rationing supply than the published version, where we imposed

$\sigma_t(p, K) = \{y \in S_{j(t)}(p) \,|\, \big(Y_{j(t)} - \{y\}\big) \subset -K\}$

as opposed to

$\sigma_t(p, K) = \{y \in S_{j(t)}(p) \,|\, p \neq 0_{\mathbb{R}^L} \Rightarrow \big(Y_{j(t)} - \{y\}\big) \cap K = \{0_{\mathbb{R}^L}\}\}.$

in the current version. The change is important to ensure existence of rationing equilibria. It requires no meaningful change in the proofs of this paper. It is sufficient to replace instances where we write " $\subset -K$" by "$\cap K \in 0_{\mathbb{R}^L}$". Moreover, for $p = 0_{\mathbb{R}^L}$ and $q > 0$, a.e. consumer is at a satiation point and the results become trivial. The proofs should distinguish the trivial case from the case $p \neq 0_{\mathbb{R}^L}$. Here, no other changes adjustments are made compared to the published version.

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1 Introduction

In general equilibrium theory it is well known that a Walrasian equilibrium may fail to exist in the presence of indivisible goods (Henry (1970)) and even the core may be empty (Shapley and Scarf (1974)).

In order to consider the presence of indivisible goods in the economy, numerous authors as Broome (1972), Mas-Colell (1977), Khan and Yamazaki (1981), Quinzii (1984) - see Bobzin (1998) for a survey - consider economies with indivisible commodities and one perfectly divisible good. All these contributions suppose that the divisible commodity satisfies overriding desirability, i.e. it is so desirable by the agents that it can replace the consumption of indivisible goods. Moreover, every agent initially owns an important quantity of this good. In such a case, the non-emptiness of the core and existence of a Walras equilibrium is then ensured.

In the model developed in Florig and Rivera (2005a) it is assumed that all the consumption goods are indivisible at individual level, but perfectly divisible at the aggregate economy. The presence of a parameter, called fiat money, which does not participate in the preferences and whose only role is to facilitate the exchange among individuals, helps us to demonstrate the existence of a competitive equilibrium called *rationing equilibrium*. Under additional assumptions on the distribution of fiat money, it can be proved that the rationing equilibrium is a Walras equilibrium. Moreover, in a parallel paper (Florig and Rivera (2005b)) we prove that the rationing equilibrium converges to a Walrasian one when the level of indivisibility converges to zero. Thus, the rationing equilibrium concept appears as a natural extension of the Walras equilibrium in the framework mentioned.

In this paper, we study welfare properties and core equivalence for rationing equilibria. For that, in our context preference relations are always locally satiated since all goods are indivisible. Konovalov (2005) shows that the standard core concepts have undesirable properties in economies with satiation. He introduced the rejective core which overcomes such drawbacks.

Using the blocking concept introduced by Konovalov (2005), we show in Proposition 1 that a rationing equilibrium cannot be blocked, whereas in Proposition 2 is proven that a rejective core allocation can be decentralized as a Walras equilibrium by an appropriate redistribution of fiat money.

With respect to the welfare analysis, we point out that at a rationing equilibrium (and for Walras one in our setting as well) it is possible that some consumers may own commodities which are worthless to them as a consumption good (or they own more than some satiation level). With indivisible goods, the value of these commodities at the equilibrium may be so small that selling them does not enable to buy more of the goods they are interested. Thus, they may waste these commodities, which may however be very useful and expensive for other agents. So the market is not as efficient as in the standard Arrow-Debreu setting (Arrow and Debreu (1954)). However, even though the standard notion of "strong Pareto optimality" fails for our equilibrium notion, this is not the case when instead we consider "weak Pareto" optimality. As e.g. Florig (2001), we use a slightly different notion of weak Pareto optimality than the one usually encountered in the literature. Using the standard weak Pareto optimality would imply that in the presence of a consumer not interested in any good, all feasible allocations are weakly Pareto optimal. We avoid this drawback.

2 Model and preliminaries

We consider the set up under which Florig and Rivera (2005a) establish equilibrium existence. We set $L \equiv \{1, \ldots, L\}$, $I \equiv \{1, \ldots, I\}$ and $J \equiv \{1, \ldots, J\}$ to denote the finite set of commodities, the finite sets of types of consumers and producers, respectively. We assume that each type $k \in I, J$ of agents consists of a continuum of identical individuals indexed by a set $T_k \subset \mathbb{R}$ of finite Lebesgue measure[1]. We set $\mathcal{I} = \cup_{i \in I} T_i$ and $\mathcal{J} = \cup_{j \in J} T_j$. Of course, $T_k \cap T_{k'} = \emptyset$ if $k \neq k'$. Given $t \in \mathcal{I}$ ($\mathcal{J}$), let

$$i(t) \in I \ (j(t) \in J)$$

be the index such that $t \in T_{i(t)}$ $(t \in T_{j(t)})$.

Each firm of type $j \in J$ is characterized by a finite production set[2] $Y_j \subset \mathbb{R}^L$ and the aggregate production set of firms of type $j \in J$ is the convex hull of $\lambda(T_j)Y_j$, which is denoted by $\mathrm{co}\lambda(T_j)Y_j$.

Every consumer of type $i \in I$ is characterized by a finite consumption set $X_i \subset \mathbb{R}^L$, an initial endowment $e_i \in \mathbb{R}^L$ and a strict preference correspondence $P_i : X_i \to X_i$.

Let $e = \sum_{i \in I} \lambda(T_i) e_i$ be the aggregate initial endowment of the economy and for $(i, j) \in I \times J$, $\theta_{ij} \geq 0$ is the share of type $i \in I$ consumers in type $j \in J$ firms. For all $j \in J$, assume that $\sum_{i \in I} \lambda(T_i)\theta_{ij} = 1$.

The initial endowment of fiat money for an individual $t \in \mathcal{I}$ is defined by $m(t)$, where $m : \mathcal{I} \to \mathbb{R}_+$ is a Lebesgue-measurable and bounded mapping.

Given all the above, an *economy* $\mathcal{E}$ is a collection

$$\mathcal{E} = \left((X_i, P_i, e_i, m)_{i \in I}, (Y_j)_{j \in J}, (\theta_{ij})_{(i,j) \in I \times J}\right),$$

an *allocation* (or consumption plan) is an element of

$$X = \left\{x \in L^1(\mathcal{I}, \cup_{i \in I} X_i) \,|\, x_t \in X_{i(t)} \text{ for a.e. } t \in \mathcal{I}\right\},$$

a *production plan* is an element of

$$Y = \left\{y \in L^1(\mathcal{J}, \cup_{j \in J} Y_j) \,|\, y_t \in Y_{j(t)} \text{ for a.e. } t \in \mathcal{J}\right\},$$

and the *feasible consumption-production plans* are elements of

$$A(\mathcal{E}) = \left\{(x, y) \in X \times Y \mid \int_{\mathcal{I}} x_t dt = \int_{\mathcal{J}} y_t dt + e\right\}.$$

[1]Without loss of generality we may assume that T_k is a compact interval of $\mathbb{R}$. In the following, we note by $\lambda(T_k)$ the Lebesgue measure of set $T_k \subseteq \mathbb{R}$. Finally, we denote by $L^1(A, B)$ the Lebesgue integrable functions from $A \subset \mathbb{R}$ to $B \subset \mathbb{R}^L$.

[2]That is, the number of admissible production plans for the firm is finite.

In the rationing equilibrium definition below we will employ *pointed cones* in $\mathbb{R}^L$. We recall that a cone $K \subseteq \mathbb{R}^L$ is pointed if $-K \cap K = \{0_{\mathbb{R}^L}\}$. In the following, let us denote by $\mathcal{C}$ the set of pointed cones in $\mathbb{R}^L$.

Given $p \in \mathbb{R}^L_+$, let us define the *supply* and *profit* of a type $j \in J$ firm as

$$S_j(p) = \text{argmax}_{y \in Y_j}\ p \cdot y \qquad\qquad \pi_j(p) = \lambda(T_j)\ \sup_{y \in Y_j}\ p \cdot y,$$

and given additionally $K \in \mathcal{C}$ we define the rationing supply (in the following simply *supply*) for a firm $t \in \mathcal{J}$ by

$$\sigma_t(p, K) = \{y \in S_{j(t)}(p) \,|\, p \neq 0_{\mathbb{R}^L} \Rightarrow \left(Y_{j(t)} - \{y\}\right) \cap K = \{0_{\mathbb{R}^L}\}\}.^3$$

For prices $(p, q) \in \mathbb{R}^L \times \mathbb{R}_+$, we denote the *budget set* of a consumer $t \in \mathcal{I}$ by

$$B_t(p, q) = \left\{x \in X_{i(t)} \,|\, p \cdot x \leq w_t(p, q)\right\},$$

where

$$w_t(p, q) = p \cdot e_{i(t)} + qm(t) + \sum_{j \in J} \theta_{i(t)j} \pi_j(p)$$

is the wealth of individual $t \in \mathcal{I}$. The set of *maximal elements* for the preference relation in the budget set for consumer $t \in \mathcal{I}$ is denoted by $d_t(p, q)$ and given that, we define the weak demand at the respective prices as[4]

$$D_t(p, q) = \limsup_{(p', q') \to (p, q)} d_t(p', q').$$

This auxiliary concept is finally employed to define our notion of *demand*, which for a cone $K \in \mathcal{C}$ and prices $(p, q) \in \mathbb{R}^L \times \mathbb{R}_+$ is defined as

$$\delta_t(p, q, K) = \{x \in D_t(p, q) \,|\, P_{i(t)}(x) - \{x\} \subset K\}.$$

Remark 1 *Intuition behind rationing equilibrium*

Following Florig and Rivera (2005a), in our model a consumer might be unable to obtain a maximal element within his budget set. Should he be unable to buy $\xi \in B_t(p, q)$ with $p \cdot \xi < w_t(p, q)$, then he could try to pay this bundle at a higher price than the market price in order to be "served first" and therefore there is some pressure on the price of the bundle ξ and its price would rise, if a non-negligible set of consumers is rationed in this sense. So at equilibrium, no consumer obtains a bundle of goods $x \in B_t(p, q)$ such that a strictly preferred bundle ξ with $p \cdot \xi < w_t(p, q)$ exists. As explained in Florig and Rivera (2005a), this notion of demand could lead to an *unstable situation* if the agents have more information than their own characteristics and the market price.

To eliminate the instability above mentioned it is however not necessary that the agents have a precise information on their trading partners, it is enough that they know which kind of net-trades are difficult to realize on the market (which is the "short" side

[3] As discussed on page one, this differs from the definition of the published version.

[4] See Rockafellar and Wets (1988) for the *limsup* definition of a correspondence.

of the market) when formulating their demand. This is summarized by the cone K as before. It is natural to consider only cones which do not contain straight lines, i.e. if a direction of net-trade is difficult to realize, the opposite direction is easy to realize. One could think of the demand for the rationing equilibrium as follows. First agents perceive the market price and the cone K and then they compute their budget set. They try to find out for which type of allocations they could find a counterpart. So an allocation is not acceptable, if there exists a preferred one in the budget set which costs less than their total wealth. Moreover, they do not accept an allocation x, if a preferred allocation x' exists which is contained in the budget set and such that $x'-x \notin K$. In fact, it should not be difficult to find a counterpart for the net-exchange $x'-x$. Alternatively think that they first accept the allocation x, but then they make another net-exchange $x'-x$ leading to x' and so on, until they are at an allocation ξ such that $P_{i(t)}(\xi)-\{\xi\} \subset K$. At this stage, obtaining a preferred allocation would require a net-exchange of a direction for which it is difficult to find a counterpart.

As for the firms, in their supply, as defined here, they do not only maximize profit as in the weak (or standard) supply, but amongst the profit maximizing production plans, they choose the one which should be the most "easy" to sell according to the cone K.

Remark 2 From Florig and Rivera (2005a), if $qm(t) > 0$ then

$$D_t(p,q) = \left\{x \in B_t(p,q) \mid p \cdot P_{i(t)}(x) \geq w_t(p,q),\ x \notin \mathrm{co}P_{i(t)}(x)\right\}.$$

If $qm(t) > 0$ then fiat money can be used as an intermediary good.

Remark 3 In the definition of demand, at least when $qm(t) > 0$ for a.e. $t \in \mathcal{I}$, we could equivalently write $(P_{i(t)}(x) - \{x\}) \cap p^{\perp} \subset K$ instead of $P_{i(t)}(x) - \{x\} \subset K$, where $p^{\perp}$ is the orthogonal to p. Indeed, preferred net trades with a negative value can anyway not exist by the definition of the weak demand, and those with a strictly positive value will be outside the budget set.

With the previous concepts, we can now define our equilibrium notions.

Definition 1 Let $(x,y,p,q) \in A(\mathcal{E}) \times \mathbb{R}^L \times \mathbb{R}_+$ and $K \in \mathcal{C}$.

We call (x,y,p,q) a *Walras equilibrium* of $\mathcal{E}$ if for a.e. $t \in \mathcal{I}$, $x_t \in d_t(p,q)$ and for a.e. $t \in \mathcal{J}$, $y_t \in S_{j(t)}(p)$.

We call (x,y,p,q,K) a *rationing equilibrium* of $\mathcal{E}$ if for a.e. $t \in \mathcal{I}$, $x_t \in \delta_t(p,q,K)$ and for a.e. $t \in \mathcal{J}$, $y_t \in \sigma_t(p,K)$.

Remark 4 Note that every Walras equilibrium is a rationing equilibrium with K being the cone generated by $P_{i(t)}(x_t) - \{x_t\}$. We refer to Florig and Rivera(2005a) for the conditions that ensure existence of these two equilibrium notions in the current framework.

On the other hand, it is well known that a Walras equilibrium may fail to exists when goods are indivisible. Mathematically this comes from the fact that in our framework the correspondence d_t is not necessarily upper semi continuous with respect to (p,q), unlike the regularized notion of it (D_t).

3 Core properties

Konovalov (2005) shows that the standard core notions have undesirable properties when preferences are satiated, which is obviously our case due the indivisibility of all goods in our setting. To overcome these undesirable properties, he proposes a new notion of blocking that is used here to study the core properties of the rationing equilibrium. Thus, we establish equivalence between rationing equilibrium allocations (which satisfy $qm(t) > 0$ a.e.) and the rejective core, and then we illustrate our result with examples.

The following definition is an straightforward extension of Konovalov (2005) rejective core to our setting. It was already used in Florig (2001) in order to establish a core equivalence result for hierarchic equilibria when the strong survival and non-satiation assumption may fail to hold.

Definition 2 The coalition $T \subset \mathcal{I}$ rejects $(x, y) \in A(\mathcal{E})$, if there exist a measurable partition U, V of T, and an allocation $(x', y') \in X \times Y$ such that the following holds

(i)

$$\int_T x'_t dt =$$

$$\int_U \left(x_t + \sum_{j \in J} \theta_{i(t)j} \int_{T_j} (y'_\tau - y_\tau) d\tau \right) dt + \int_V \left(e_{i(t)} + \sum_{j \in J} \theta_{i(t)j} \widetilde{y}_j(V) \right) dt,$$

with $\widetilde{y}_j(V) = \int_V y'_\tau d\tau$,

(ii) $x'_t \in P_{i(t)}(x_t)$ for a.e. $t \in T$.

The rejective core $\mathcal{RC}(\mathcal{E})$ of $\mathcal{E}$ is the set of $(x, y) \in A(\mathcal{E})$ that cannot be rejected by a non-negligible coalition.

Remark 5 Florig (2001) gives an interpretation of Konovalov's (2005) rejective core where each consumer, or here rather each set of consumers with strictly positive Lebesgue measure, can control his "share" of the production set. Thus, a proposal (x, y) could be rejected by group $V \cup U$ with the following argument: even if $T \setminus (U \cup V)$ could achieve to implement the proposal with some of us, i.e. with U, then once this is realized, we would change the outcome on the part of the economy we control from x to x'.

Proposition 1 *Let (x, y, p, q, K) be a rationing equilibrium such that for a.e. $t \in \mathcal{I}$, $qm(t) > 0$, then $(x, y) \in \mathcal{RC}(\mathcal{E})$.*

Proof. We assume $p \neq 0_{\mathbb{R}^L}$ since otherwise the Proposition is trivial. Let $T \subset \mathcal{I}$ with $\lambda(T) > 0$, U, V a measurable partition of T and $(x', y') \in A(\mathcal{E})$ such that conditions (i)-(ii) of Definition 2 hold.

By condition (ii) and Remark 2 we have that for a.e. $t \in T$, $p \cdot x'_t \geq w_t(p, q)$, and then, considering that $qm(t) > 0$ for a.e. $t \in \mathcal{I}$, we conclude that for a.e. $t \in V$

$$p \cdot \left[e_{i(t)} + \sum_{j\in J} \theta_{i(t)j} \widetilde{y}_j(V) \right] < w_t(p,q).$$

On the other hand, by profit maximization we have that for a.e. $t \in U$

$$p \cdot \left[\sum_{j\in J} \theta_{i(t)j} \int_{T_j} (y'_\tau - y_\tau) d\tau \right] \leq 0,$$

and therefore

$$p \cdot \left[x_t + \sum_{j\in J} \theta_{i(t)j} \int_{T_j} (y'_\tau - y_\tau) d\tau \right] \leq w_t(p,q).$$

Hence, if $\lambda(V) > 0$ we would have

$$p \cdot \int_T x'_t dt \geq \int_T w_t(p,q) dt >$$

$$p \cdot \left[\int_U \left(x_t + \sum_{j\in J} \theta_{i(t)j} \int_{T_j} (y'_\tau - y_\tau) d\tau \right) dt + \int_V \left(e_{i(t)} + \sum_{j\in J} \theta_{i(t)j} \widetilde{y}_j(V) \right) dt \right],$$

contradicting condition (i). So we have $\lambda(V) = 0$ and we must have

$$p \cdot \int_U x'_t dt = p \cdot \left[\int_U \left(x_t + \sum_{j\in J} \theta_{i(t)j} \int_{T_j} (y'_\tau - y_\tau) d\tau \right) dt \right].$$

By Remark 2, for a.e. $t \in U$, $p \cdot (x'_t - x_t) \geq 0$ and since by profit maximization $p \cdot \left(\sum_{j\in J} \theta_{i(t)j} \int_{T_j} (y'_\tau - y_\tau) d\tau \right) \leq 0$, we must have for a.e. $t \in U$, $p \cdot (x'_t - x_t) = 0$ and $p \cdot \left(\sum_{j\in J} \theta_{i(t)j} \int_{T_j} (y'_\tau - y_\tau) d\tau \right) = 0$. Therefore by definition of demand and supply, for a.e. $t \in U$, $(x'_t - x_t) \in K$ and $\sum_{j\in J} \theta_{i(t)j} \int_{T_j} (y'_\tau - y_\tau) d\tau \cap K \in \{0_{\mathbb{R}^L}\}$ and integrating over U, we have $\int_U (x'_t - x_t) dt \in K$ and

$$\int_U \left(\sum_{j\in J} \theta_{i(t)j} \int_{T_j} (y'_\tau - y_\tau) d\tau \right) dt \cap K \in \{0_{\mathbb{R}^L}\}.$$

Now, since

$$K \ni \int_U (x'_t - x_t) dt = \int_U \left(\sum_{j\in J} \theta_{i(t)j} \int_{T_j} (y'_\tau - y_\tau) d\tau \right) dt,$$

we have $\int_U (x'_t - x_t) dt = 0_{\mathbb{R}^L}$. Since for a.e. $t \in U$, $(x'_t - x_t) \in K$, this implies that for a.e. $t \in U$, $x'_t = x_t$, which is a contradiction with condition (ii) from Definition 2. □

With production, the absence of local non-satiation entails the possible existence of rejective core allocations that can not be decentralized. This is due to the fact that a consumer at a satiation point does not care whether a firm he entirely owns chooses a profit maximizing production plan or not. This could be overcome by a refinement of profit maximization as in Florig (2001). Instead, we show that without a production sector every rejective core allocation can be decentralized.

Proposition 2 *Suppose $J = \emptyset$ (exchange economy). Then, for every $x \in \mathcal{RC}(\mathcal{E})$ there exists $(p, m') \in \mathbb{R}^L \setminus \{0\} \times L^1(\mathcal{I}, \mathbb{R}_{++})$ such that $(x, p, q = 1)$ is a Walras equilibrium with money of the economy $\mathcal{E}$ when replacing m by m'.*

Proof. Let $x \in \mathcal{RC}(\mathcal{E})$. Since the number of types is finite and the consumption sets are finite, we can define a finite set of consumer types $A \equiv \{1, \dots, A\}$ satisfying the following

(i) $(T_a)_{a\in A}$ is a finer partition of $\mathcal{I}$ than $(T_i)_{i\in I}$,

(ii) for every $a \in A$, there exists x_a such that for every $t \in T_a$, $x_t = x_a$.

Set

$$H_a = \lambda(T_{i(a)}) \left(\mathrm{co}P_{i(a)}(x_a) - \{x_a\}\right), \; G_a = \lambda(T_{i(a)}) \left(\mathrm{co}P_{i(a)}(x_a) - \{e_{i(a)}\}\right),$$

$$\mathcal{K} = \mathrm{co} \bigcup_{a\in A} (G_a \cup H_a).$$

If $\mathcal{K} = \emptyset$, then every consumer is satiated and the problem becomes trivial. So let us assume $\mathcal{K} \neq \emptyset$. As a first step of the demonstration we will prove that $0 \notin \mathcal{K}$. Otherwise there exist $(\sigma_a) \in [0,1]^A$ and $(\mu_a) \in [0,1]^A$ with $\sum_{a\in A}(\sigma_a + \mu_a) = 1$ and $\xi_a \in \mathrm{co}P_{i(a)}(x_a)$ for all $a \in A$, such that

$$\sum_{a\in A} (\sigma_a \lambda(T_a)(\xi_a - x_a) + \mu_a \lambda(T_a)(\xi_a - e_a)) = 0.$$

Now let $T \subset \mathcal{I}$ such that for each $a \in A$, $\lambda(T \cap T_a) = (\sigma_a + \mu_a)\lambda(T_a)$. Thus there exists a measurable partition U, V of T (those for which $\sigma_a, \; \mu_a > 0$) and $\xi \in X$ such that for a.e. $t \in T$ and $\xi_t \in P_{i(t)}(x_t)$ and for all $a \in A$

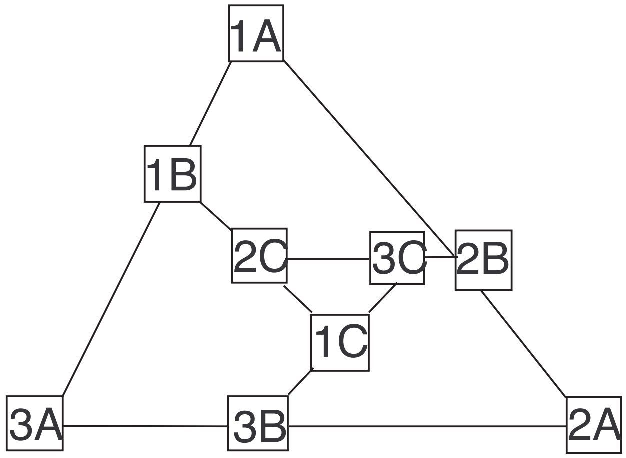
$$\lambda(U \cap T_a) = \frac{1}{2}\sigma_a \lambda(T_a),$$

$$\lambda(V \cap T_a) = \frac{1}{2}\mu_a \lambda(T_a),$$

and (it is easy to check)

$$\int_T \xi_t dt = \int_U x_t dt + \int_V e_t dt.$$

Thus, coalition T can reject x and therefore $0 \notin \mathcal{K}$. Finally, since $\mathcal{K}$ is compact there exists $p \in \mathbb{R}^L \setminus \{0_{\mathbb{R}^L}\}$ and $\varepsilon > 0$ such that $0 < \varepsilon < \min p \cdot \mathcal{K}$. For every $a \in A$, given t such that $i(t) = a$ define $m'(t) = p \cdot (x_t - e_a) + \varepsilon/2$ and set $q = 1$. Thus $m'(t) > 0$. Then, of course for every $t \in \mathcal{I}$, $p \cdot x_t < p \cdot e_{i(t)} + qm'(t) < \min p \cdot P_t(x_t)$, which ends the proof ($m'(t) = m(t)$ for $t \in \mathcal{I} \setminus T$). □



Remark 6 *Rationing equilibria without money may be rejected.*

In the equilibrium definition we did not impose that the price of fiat money is positive. The present example shows that a positive price of fiat money is needed in order to ensure that an equilibrium allocation is in the rejective core. Consider an exchange economy with three types of consumers (with $\lambda(T_1) = \lambda(T_2) = \lambda(T_3)$) and two commodities: for all $i \in I$, $X_i = \{0, 1, \ldots, 5\} \times \{0, 1, \ldots, 5\}$, $u_1(x) = -x^1 - x^2$, $u_2(x) = -\|x - (1,1)\|_1$, $u_3(x) = -\|x - (0,1)\|_1$, $e_1 = (0,4)$, $e_2 = (0,0)$, $e_3 = (1,0)$. The type symmetric allocation $x_1 = (0,0)$, $x_2 = (1,2)$, $x_3 = (0,2)$ is a rationing equilibrium with $p = q = 0$, $K = \{t(0,-1) \mid t \geq 0\}$. However this rationing equilibrium is not in the rejective core since the players of type 2 and 3 may reject with the type symmetric allocation $\xi_1 = (0,2), \xi_2 = (1,1)$ and $\xi_3 = (0,1)$.

To end this section, we use an example from Shapley and Scarf (1974) to illustrate some facts mentioned in this section.

Example 1 Shapley and Scarf (1974) gave the following example in order to show that the core may be empty when commodities are indivisible. We consider an economy with three types of agents $I = \{1,2,3\}$ nine commodities $L = \{1_A, 1_B, 1_C, \ldots, 3_C\}$, commodity sets $X_i = \{0,1\}^9$ and concave utility functions for $i \in I$

$$u_i(x) = \max\{2\min\{x_{i_A}, x_{i+1_A}, x_{i+1_B}\}; \min\{x_{i_C}, x_{i+2_B}, x_{i+2_C}\}\}.$$

The indices are module 3. Initial endowments are $e_i = (e_{ih}) \in X_i$ with $e_{ih} = 1$ if and only if $h \in \{i_A, i_B, i_C\}$.

The picture below illustrates endowments and preferences. Each consumer would like to have three commodities on a straight line containing only one of his commodities. The best bundle is to own a long line containing his commodity i_A and $i+1_B, i+1_A$ and the second best would be to own a short line containing his commodity i_C and $i+2_B, i+2_C$.

If there is only one agent per type this reduces indeed to Shapley and Scarf's (1974) setting. In this case, at any feasible allocation for some $i \in I$, agent i obtains utility zero and agent $i+2$ at most utility one. However, if they form a coalition it is possible to give utility one to i and two to $i+2$. Thus, the core is empty.

With an even number of agents per type or a continuum of measure one per type the weak and the rejective core correspond to the allocations such that half of the consumers of type i consume $x_{ih} = 1$ for all $h \in \{i_A, i+1_A, i+1_B\}$ and the other half consumes $x_{ih} = 1$ for all $h \in \{i_C, i+2_B, i+2_C\}$. So every consumer obtains at least his second best allocation. It is not possible to block an allocation in the sense that all consumers who block are strictly better off. Indeed, they would all need to obtain their best allocation and this is not feasible for any group. To see that this is the only allocation in the core, note that at any other allocation at least one consumer say a consumer of type 1 (or a non-negligible group of a given type) would necessarily get an allocation which yields zero utility. Then by feasibility, a consumer of type 3 (or a non-negligible group of type 3) obtain only their second best choice. The consumer of type 1 can propose the commodities $1_A, 1_B$ in exchange for $3_B, 3_C$ making everybody strictly better off.

Allocations in the core are supported by a uniform distribution of fiat money $m_i = m > 0$ for all $i \in I$ and the price $p = (2, 1, 1, 2, 1, 1, 2, 1, 1)$, $q = 1/m$. Thus, a Walras equilibrium with money does not exist for a uniform distribution of paper money. A rationing equilibrium, however, exists. If half of each type obtains one unit of paper money and the other half strictly less than one unit, then the core allocation is a Walras equilibrium allocation with the same price p and $q = 1$.

4 Welfare Analysis

As we mentioned in the introduction, a rationing equilibrium will not necessarily be a strong Pareto optimum[5]. This comes from the fact that in presence of indivisible goods some consumers may own commodities that are worthless to them as a consumption good since the value of these commodities may be so small at the equilibrium that selling them does not enable to buy more of the indivisible goods they are interested.

If the preference relation of at least one consumer is empty valued for all allocations then any feasible allocation would be a weak Pareto optimum. Of course we could have pathologic weak optima even under less extreme circumstances. This motivates the following definition which was also used in Florig (2001).

Definition 3 A collection $(x, y) \in A(\mathcal{E})$ is a Pareto optimum if there does not exist $(x', y') \in A(\mathcal{E})$ and a non-negligible set $T \subset \mathcal{I}$ such that for a.e. $t \in T$, $x'_t \in P_{i(t)}(x_t)$ and for a.e. $t \in \mathcal{I}$, $x'_t \neq x_t$ if and only if $t \in T$.

Remark 7 Note that if in an exchange economy an allocation x is not Pareto optimal, then it could be rejected by the coalition of consumers who under x' obtain a different allocation. All consumers could first execute x and in a second step switch to x'. In a

[5]We recall that a feasible allocation is a *strong Pareto optimum* if there does not exists another feasible allocation which is preferred to the original one by all and strictly preferred for some consumers; the allocation is a *weak Pareto optimum* if there does not exists another feasible allocation which is strictly preferred by all.

production economy, it would be possible that all production sets belong to consumers who are satiated. Then, an allocation could be in the rejective core, but fail to fulfill our optimality criterion. Indeed, switching to an allocation which is better for the non satiated consumers might require production, but the satiated consumers may have no incentive to implement it.

Proposition 3 First Welfare Theorem.
Every rationing equilibrium is a Pareto optimum.

Proof. We assume $p \neq 0_{\mathbb{R}^L}$ since otherwise the Proposition is trivial. Let (x, y, p, q, K) be a rationing equilibrium and $(x', y') \in A(\mathcal{E})$ Pareto dominating (x, y), with T the non-negligible set from Definition 3. From feasibility we already know that

$$e = \int_{\mathcal{I}} x'_t dt - \int_{\mathcal{J}} y'_t dt = \int_{\mathcal{I}} x_t - \int_{\mathcal{J}} y_t dt.$$

Therefore,

$$\int_{\mathcal{J}} (y'_t - y_t)\, dt = \int_{\mathcal{I}} (x'_t - x_t)\, dt,$$

and since for a.e. $t \in \mathcal{I}$, $x_t \in \delta_t(p, q, K)$ we have for a.e. $t \in \mathcal{I}$ with $x_t \neq x'_t$, $p \cdot (x'_t - x_t) \geq 0$ and thus $p \cdot \int_{\mathcal{I}} (x'_t - x_t)\, dt \geq 0$.

By profit maximization $p \cdot \int_{\mathcal{J}} (y'_t - y_t)\, dt \leq 0$. Therefore $p \cdot \int_{\mathcal{I}} (x'_t - x_t)\, dt = p \cdot \int_{\mathcal{J}} (y'_t - y_t)\, dt = 0$ and moreover $p \cdot (x'_t - x_t) = 0$. In consequence for a.e. $t \in \mathcal{I}$ with $x_t \neq x'_t$, $(x'_t - x_t) \in K$ and $\int_{\mathcal{I}} (x'_t - x_t) dt \in K$. By the supply definition we have now for a.e. $t \in \mathcal{J}$, $p \cdot \int_{\mathcal{J}} (y'_t - y_t) dt = 0$ and therefore $(y'_t - y_t) \cap K \in \{0_{\mathbb{R}^L}\}$ and $\int_{\mathcal{J}} (y'_t - y_t) dt \cap K \in \{0_{\mathbb{R}^L}\}$. Thus

$$K \ni \int_{\mathcal{I}} (x'_t - x_t) = \int_{\mathcal{J}} (y'_t - y_t)\, ,$$

which implies $\int_{\mathcal{I}} (x'_t - x_t)\, dt = 0_{\mathbb{R}^L}$ and since for a.e. $t \in \mathcal{I}$ with $x_t \neq x'_t$ we had that $(x'_t - x_t) \in K$, and since K is a pointed cone, we finally conclude that for a.e. $t \in \mathcal{I}$, $x_t = x'_t$, which is a contradiction. □

For decentralization of Pareto optima we will focus on exchange economies for the same reasons as for the core decentralization. Again an approach similar to Florig (2001) should allow for decentralizing Pareto optima.

Proposition 4 Second Welfare Theorem.
Let $\mathcal{E}$ be an economy with $J = \emptyset$ (exchange economy). If x is a Pareto optimum, then there exists $p \in \mathbb{R}^L \setminus \{0\}$ and $e' \in X$ such that (x, p) is a Walras equilibrium of $\mathcal{E}'$ which is obtained from $\mathcal{E}$, replacing the initial endowment e by e'.

Proof. For all $t \in \mathcal{I}$ set $e'_{i(t)} = x_{i(t)}$. Since the number of types is finite and the consumption sets are finite, we can define a finite set of consumer types $A \equiv \{1, \ldots, A\}$ satisfying the following

(i) $(T_a)_{a \in A}$ is a finer partition of $\mathcal{I}$ than $(T_i)_{i \in I}$,

(ii) for every $a \in A$, there exists x_a such that for every $t \in T_a$, $x_t = x_a$.

Define

$$H_a = \lambda(T_a)\left(\text{co}P_a(x_a) - \{x_a\}\right),$$

and

$$\mathcal{H} = \text{co}\bigcup_{a \in A} H_a.$$

If $\mathcal{H} = \emptyset$, then everybody is satiated and any p would do. Let us assume we are in the non trivial case $\mathcal{H} \neq \emptyset$. Note that $0_{\mathbb{R}^L} \notin \mathcal{H}$. Otherwise there exist $(\sigma_a) \in [0,1]^A$ with $\sum_{a \in A} \sigma_a = 1$ and $\xi_a \in \text{co}P_a(x_a)$ for all $a \in A$, such that $\sum_{a \in A} \sigma_a \lambda(T_a)(\xi_a - x_a) = 0$. Thus there exists $\xi \in X$ such that for all $a \in A$

$$\lambda(\{t \in T_a \mid \xi_t \in P_{i(t)}(x_t)\}) = \sigma_a \lambda(T_a),$$

and

$$\lambda(\{t \in T_a \mid \xi_t = x_t\}) = (1 - \sigma_a)\lambda(T_a),$$

contradicting Pareto optimality of x.

Since the consumption sets are all compact, we have that for all $a \in A$, H_a is compact. Since A is finite, we have that $\mathcal{H}$ is compact. Thus there exists $p \in \mathbb{R}^L \setminus \{0_{\mathbb{R}^L}\}$ and $\varepsilon > 0$ such that for all $z \in \mathcal{H}$, $p \cdot z > \varepsilon$. Hence for a.e. $t \in \mathcal{I}$,

$$P_{i(t)}(x_t) \cap \{\xi \in X_{i(t)} \mid p \cdot \xi \leq p \cdot x_t + \varepsilon\} = \emptyset.$$

So (x, p) is indeed a Walras equilibrium of $\mathcal{E}'$. Setting $q > 0$ such that for all $i \in I$, $qm(t) < \varepsilon/2$ with $i(t) = i$, (x, p, q) would also be a Walras equilibrium with a positive value of fiat money, which ends the demonstration. □

Remark 8 Under the assumptions of the previous proposition, we could also decentralize any Pareto optimum x by collecting taxes $\tau_t = p \cdot (x_t - e_{i(t)}) + m(t)$ from agent $t \in \mathcal{I}$ payable in monetary units. Then, x becomes an equilibrium together with $q = 1$ and p as in the previous proof.

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